

Towards a Theory of Conservative Computing

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We extend the notion of conservativeness, given by Fredkin and Toffoli in 1982, to generic gates whose input and output lines may assume a finite number d of truth values. A physical interpretation of conservativeness in terms of conservation of the energy associated to the data used during the computation is given. Moreover, we define *conservative computations*, and we show that they naturally induce a new NP-complete decision problem. Finally, we present a framework that can be used to explicit the movement of energy occurring during a computation, and we provide a quantum implementation of the primitives of such framework using creation and annihilation operators on the Hilbert space \mathbb{C}^d , where d is the number of energy levels considered in the framework.

1. INTRODUCTION

Conservative logic has been introduced in (Fredkin and Toffoli, 1982) as a mathematical model that allows one to describe computations which reflect some properties of microdynamical laws of physics, such as reversibility and conservation of the internal energy of the physical system used to perform the computations. The model is based on the so-called Fredkin gate, a three-input/three-output Boolean gate originally introduced by Petri in (Petri, 1967), whose input/output map $FG : \{0, 1\}^3 \rightarrow \{0, 1\}^3$ associates any input triple (x_1, x_2, x_3) with its corresponding output triple (y_1, y_2, y_3) as follows:

$$y_1 = x_1, \quad y_2 = (\neg x_1 \wedge x_2) \vee (x_1 \wedge x_3), \quad y_3 = (x_1 \wedge x_2) \vee (\neg x_1 \wedge x_3)$$

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The Fredkin gate is *functionally complete* for the Boolean logic: by fixing $x_3 = 0$ we get $y_3 = x_1 \wedge x_2$, whereas by fixing $x_2 = 1$ and $x_3 = 0$ we get $y_2 = \neg x_1$. A useful point of view is that the Fredkin gate behaves as a *conditional switch*: that is, $\text{FG}(1, x_2, x_3) = (1, x_3, x_2)$ and $\text{FG}(0, x_2, x_3) = (0, x_2, x_3)$ for every $x_2, x_3 \in \{0, 1\}$. In other words, the first input line can be considered as a control line whose value determines whether the input values x_2 and x_3 have to be exchanged or not.

According to Fredkin and Toffoli (1982), *conservativeness* is usually modeled by the property that the output patterns of the involved gates are always a permutation of the patterns given in input. Let us stress that this does not mean that a fixed permutation is applied to every possible input pattern; on the contrary, the applied permutation depends on the input pattern. Here we just mention the fact that every permutation can be written as a composition of transpositions. Hence, not only the Fredkin gate can be used to build an appropriate circuit to perform any given conservative computation (and thus it is universal also in this sense with respect to conservative computations), but it is also the most elementary conceivable operation that can be used to describe conservative computations. In this paper we will propose some analogous elementary operations with respect to our notion of conservativeness.

The Fredkin gate is also *reversible*, that is, FG is a bijective map on $\{0, 1\}^3$. Notice that conservativeness and reversibility are two independent notions: a gate can satisfy both properties, only one of them, or none. Since every reversible gate computes a bijective map between input and output patterns, and every conservative gate produces permutations of its input patterns, it follows that they must necessarily have the same number of input and output lines.

In this paper we extend the notion of conservativeness to generic gates whose input and output lines may assume a finite number d of truth values, and we derive some properties which are satisfied by conservative gates. By associating equispaced energy levels to the truth values, we show that our notion of conservativeness corresponds to the energy conservation principle applied to the data which are manipulated during the computation. Let us stress that we are *not* saying that the entire energy used to perform the computation is preserved, or that the computing device is a conservative physical system. In particular, we do not consider the energy needed to transform the input values into output values, that is, the energy needed to *perform* the computation.

Successively, we introduce the notion of *conservative computation*, based on gates which are able to store some finite amount of energy and to reuse it during the computation. We show that the decision problem to determine whether a given computation can be performed in a conservative way through a gate which is able to store at most C units of energy is NP-complete.

Finally, we introduce a framework that allows one to visualize the movement of energy occurring during a computation performed by a generic gate. The

framework is based on some primitive operators that conditionally move one unit of energy between any two given input/output lines of the gate. Using creation and annihilation operators on the Hilbert space \mathbb{C}^d , we show a quantum realization of these non-unitary conditional movement operators.

2. CONSERVATIVENESS

Our notion of conservativeness, and the framework we will introduce, are based on many-valued logics. These are extensions of the classical Boolean logic which are widely used to manage incomplete and/or uncertain knowledge. Different approaches to many-valued logics have been considered in literature: for an overview, see (Rescher, 1969; Rosser and Turquette, 1952). However, here we are not interested into the study of syntactical or algebraic aspects of many-valued logics; we just define some gates whose input and output lines may assume “intermediate” truth values, such as the gates defined in (Cattaneo *et al.*, 2002).

For every integer $d \geq 2$, we consider the finite set $L_d = \{0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}, 1\}$ of truth values; 0 and 1 denote falsity and truth, respectively, whereas the other values of L_d indicate different degrees of indefiniteness. As usually found in literature, we will use L_d both as a set of truth values and as a numerical set equipped with the standard order relation on rational numbers.

An n -input/ m -output d -valued function (also called an (n, m, d) -function for short) is a map $f : L_d^n \rightarrow L_d^m$. Analogously, an (n, m, d) -gate and an (n, m, d) -circuit are devices that compute (n, m, d) -functions. A gate is considered as a primitive operation, that is, it is assumed that a gate cannot be decomposed into simpler parts. On the other hand, a circuit is composed by layers of gates, where any two gates G_1 and G_2 of the same layer satisfy the property that no output line of G_1 is connected to any input line of G_2 .

Let us consider the set $\mathcal{E}_d = \{\varepsilon_0, \varepsilon_{\frac{1}{d-1}}, \varepsilon_{\frac{2}{d-1}}, \dots, \varepsilon_{\frac{d-2}{d-1}}, \varepsilon_1\} \subseteq \mathbb{R}$ of real values; for exposition convenience, we can think to such quantities as energy values. To each truth value $v \in L_d$ we associate the energy level ε_v ; moreover, let us assume that the values of \mathcal{E}_d are all positive, equispaced, and ordered according to the corresponding truth values: $0 < \varepsilon_0 < \varepsilon_{\frac{1}{d-1}} < \dots < \varepsilon_{\frac{d-2}{d-1}} < \varepsilon_1$. If we denote by δ the gap between two adjacent energy levels then the following holds:

$$\varepsilon_v = \varepsilon_0 + \delta(d - 1)v, \quad \forall v \in L_d \tag{1}$$

Notice that it is not required that $\varepsilon_0 = \delta$.

Now, let $\underline{x} = (x_1, \dots, x_n) \in L_d^n$ be an input pattern for an (n, m, d) -gate. We define the amount of energy associated to \underline{x} as $E_n(\underline{x}) = \sum_{i=1}^n \varepsilon_{x_i}$, where $\varepsilon_{x_i} \in \mathcal{E}_d$ is the amount of energy associated to the i -th element x_i of the input pattern. Let us remark that the map $E_n : L_d^n \rightarrow \mathbb{R}^+$ is indeed a family of mappings parameterized by n , the size of the input. Analogously, for an output pattern $\underline{y} \in L_d^m$ we define

the associated amount of energy as $E_m(\underline{y}) = \sum_{i=1}^m \varepsilon_{y_i}$. We can now define a conservative gate as follows.

Definition 2.1. An (n, m, d) -gate, described by the function $G : L_d^n \rightarrow L_d^m$, is *conservative* if the following condition holds:

$$\forall \underline{x} \in L_d^n, \quad E_n(\underline{x}) = E_m(G(\underline{x})) \quad (2)$$

Notice that it is not required that the gate has the same number of input and output lines, as it happens with the reversible and conservative gates considered in (Cattaneo *et al.*, 2002, in press; Fredkin and Toffoli, 1982).

Using relation (1), Equation (2) can also be written as:

$$\frac{\varepsilon_0 n}{\delta(d-1)} + \sum_{i=1}^n x_i = \frac{\varepsilon_0 m}{\delta(d-1)} + \sum_{j=1}^m y_j$$

Hence, when $n = m$ (as it happens, for example, with reversible gates) conservativeness reduces to the conservation of the sum of truth values given in input, as in weak conservativeness introduced in (Cattaneo *et al.*, 2002). In the Boolean case this is equivalent to requiring that the number of 1s given in input is preserved, as in the original notion of conservativeness given in (Fredkin and Toffoli, 1982).

An interesting remark is that conservativeness entails an upper and a lower bound to the ratio $\frac{m}{n}$ of the number of output lines versus the number of input lines of a gate. In fact, the maximum amount of energy that can be associated to an input pattern is $\sum_{i=1}^n \varepsilon_1 = n \varepsilon_1$, whereas the minimum amount of energy that can be associated to an output pattern is $\sum_{i=1}^m \varepsilon_0 = m \varepsilon_0$. Clearly, if it holds $n \varepsilon_1 < m \varepsilon_0$ then the gate cannot produce any output pattern in a conservative way. As a consequence, it must hold $\frac{m}{n} \leq \frac{\varepsilon_1}{\varepsilon_0}$. Analogously, if we consider the minimum amount of energy $n \varepsilon_0$ that can be associated to an input pattern \underline{x} and the maximum amount of energy $m \varepsilon_1$ that can be associated to an output pattern \underline{y} , it clearly must hold $n \varepsilon_0 \leq m \varepsilon_1$, that is $\frac{m}{n} \geq \frac{\varepsilon_0}{\varepsilon_1}$. Summarizing, we have the bounds $\frac{\varepsilon_0}{\varepsilon_1} \leq \frac{m}{n} \leq \frac{\varepsilon_1}{\varepsilon_0}$, that is, for a conservative gate (or circuit) the number m of output lines is constrained to grow linearly with respect to the number n of input lines.

A natural question is whether we can compute all functions in a conservative way. Let us consider the Boolean case. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ be a non-necessarily conservative function, and let us define the following quantities:

$$O_f = \max \left\{ 0, \max_{\underline{x} \in \{0,1\}^n} \{E_m(f(\underline{x})) - E_n(\underline{x})\} \right\}$$

$$Z_f = \max \left\{ 0, \max_{\underline{x} \in \{0,1\}^n} \{E_n(\underline{x}) - E_m(f(\underline{x}))\} \right\}$$

Informally, O_f (resp., Z_f) is the maximum number of 1s (resp., 0s) in the output pattern that should be converted to 0 (resp., 1) in order to make the computation conservative. This means that if we use a gate G_f with $n + O_f + Z_f$ input lines

and $m + O_f + Z_f$ output lines then we can compute f in a conservative way as follows:

$$G_f(\underline{x}, \underline{1}_{O_f}, \underline{0}_{Z_f}) = (f(\underline{x}), \underline{1}_{w(\underline{x})}, \underline{0}_{z(\underline{x})})$$

where $\underline{1}_k$ (resp., $\underline{0}_k$) is the k -tuple consisting of all 1s (resp., 0s), and the pair $(\underline{1}_{w(\underline{x})}, \underline{0}_{z(\underline{x})}) \in \{0, 1\}^{O_f+Z_f}$ is such that $w(\underline{x}) = O_f + E_n(\underline{x}) - E_m(f(\underline{x}))$ and $z(\underline{x}) = Z_f - E_n(\underline{x}) + E_m(f(\underline{x}))$.

As we can see, we use some additional input (resp., output) lines in order to provide (resp., remove) the required (resp., exceeding) energy that allows G_f to compute f in a conservative way. It is easy to see that the same trick can be applied to generic d -valued functions $f : L_d^n \rightarrow L_d^m$; instead of the number of missing or exceeding 1s, we just compute the missing or exceeding number of energy units, and we provide an appropriate number of additional input and output lines.

3. CONSERVATIVE COMPUTATIONS

Let us now introduce the notion of *conservative computation*. Let $G : L_d^n \rightarrow L_d^m$ be the function computed by an (n, m, d) -gate. Moreover, let $S_{in} = \langle \underline{x}_1, \underline{x}_2, \dots, \underline{x}_k \rangle$ be a sequence of elements from L_d^n to be used as input patterns for the gate, and let $S_{out} = \langle G(\underline{x}_1), G(\underline{x}_2), \dots, G(\underline{x}_k) \rangle$ be the corresponding sequence of output patterns from L_d^m . Let us consider the quantities $e_i = E_n(\underline{x}_i) - E_m(G(\underline{x}_i))$ for all $i \in \{1, 2, \dots, k\}$; note that, without loss of generality, by an appropriate rescaling we may assume that all e_i s are integer values. We say that the *computation* of S_{out} , obtained starting from S_{in} , is *conservative* if the following condition holds:

$$\sum_{i=1}^k e_i = \sum_{i=1}^k E_n(\underline{x}_i) - \sum_{i=1}^k E_m(G(\underline{x}_i)) = 0$$

This condition formalizes the requirement that the total energy provided by *all* input patterns of S_{in} is used to build all output patterns of S_{out} . Of course, it may happen that $e_i > 0$ or $e_i < 0$ for some $i \in \{1, 2, \dots, k\}$. In the former case, the gate has an excess of energy that should be dissipated into the environment after the production of the value $G(\underline{x}_i)$, whereas in the latter case the gate does not have enough energy to produce the desired output pattern. Since we want to avoid these situations, we assume to perform computations through gates which are equipped with an internal *accumulator* (also *storage unit*) which is able to store a maximum amount C of energy units. We call C the *capacity* of the gate. The amount of energy contained into the internal storage unit at a given time can thus be used during the next computation step if the energy of the output pattern that must be produced is greater than the energy of the corresponding input pattern.

If the output patterns $G(\underline{x}_1), G(\underline{x}_2), \dots, G(\underline{x}_k)$ are computed exactly in this order then, assuming that the computation starts with no energy stored into the

gate, it is not difficult to see that $st_1 := e_1$, $st_2 := e_1 + e_2$, \dots , $st_k := e_1 + e_2 + \dots + e_k$ is the *sequence of the amounts of energy stored* into the gate during the computation of S_{out} . We say that a given conservative computation is *C-feasible* if $0 \leq st_i \leq C$ for all $i \in \{1, 2, \dots, k\}$. Notice that for conservative computations it always holds $st_k = 0$.

In some cases, the order with which the output patterns of S_{out} are computed does not matter. We can thus consider the following problem: Given an (n, m, d) -gate that computes the map $G : L_d^n \rightarrow L_d^m$, an input sequence $\underline{x}_1, \dots, \underline{x}_k$ and the corresponding output sequence $G(\underline{x}_1), \dots, G(\underline{x}_k)$, is there a permutation $\pi \in S_k$ (the symmetrical group of order k) such that the computation of $G(\underline{x}_{\pi(1)}), G(\underline{x}_{\pi(2)}), \dots, G(\underline{x}_{\pi(k)})$ is *C-feasible*? This is a decision problem, whose relevant information is entirely provided by the values e_1, e_2, \dots, e_k , which can be formally stated as follows.

Problem 3.1. NAME: CONSCOMP.

- INSTANCE: a set $\mathcal{E} = \{e_1, e_2, \dots, e_k\}$ of integer numbers such that $e_1 + e_2 + \dots + e_k = 0$, and an integer number $C > 0$.
- QUESTION: is there a permutation $\pi \in S_k$ such that $\forall i \in \{1, 2, \dots, k\}$

$$0 \leq \sum_{j=1}^i e_{\pi(j)} \leq C? \quad (3)$$

The CONSCOMP problem can be obviously solved by trying every possible permutation π from S_k . However, this procedure requires an exponential time with respect to k , the length of the computation. A natural question is whether it is possible to give the correct answer in polynomial time. With the following theorem we show that the CONSCOMP problem is NP-complete. As it is well known (Garey and Johnson, 1979), this means that if there would exist a polynomial time algorithm that solves the problem then we could immediately conclude that the two complexity classes P and NP coincide, a very unlikely situation.

Theorem 3.1. CONSCOMP is NP-complete.

Proof: CONSCOMP is clearly in NP, since a permutation $\pi \in S_k$ has linear length and verifying whether π is a solution can be done in polynomial time. In order to conclude that CONSCOMP is NP-complete, let us show a polynomial reduction from PARTITION, which is a well-known NP-complete problem (Garey and Johnson, 1979, page 47).

Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integer numbers, and let $m = \sum_{i=1}^k a_i$. The set A is a positive instance of PARTITION if and only if there exists a set $A' \subseteq A$ such that $\sum_{a \in A'} a = \frac{m}{2}$. If m is odd then A is certainly a negative

instance, and we can associate it to any negative instance of CONSCOMP. On the other hand, if m is even we build the corresponding instance (\mathcal{E}, C) of CONSCOMP by putting $C = \frac{m}{2}$ and $\mathcal{E} = \{e_1, e_2, \dots, e_k, e_{k+1}, e_{k+2}\}$, where $e_i = -a_i$ for all $i \in \{1, 2, \dots, k\}$ and $e_{k+1} = e_{k+2} = \frac{m}{2}$. It is immediately seen that this construction can be performed in polynomial time.

We claim that A is a positive instance of PARTITION if and only if (\mathcal{E}, C) is a positive instance of CONSCOMP. In fact, let us assume that A is a positive instance of PARTITION. Then there exists a set $A' \subseteq A$ such that $\sum_{a \in A'} a = \frac{m}{2}$, and the corresponding negative elements of \mathcal{E} constitute a subset \mathcal{E}' such that $\sum_{e \in \mathcal{E}'} e = -\frac{m}{2}$. We build a permutation $\pi \in S_k$ by selecting first the element e_{k+1} followed by the elements of \mathcal{E}' (chosen with any order), and then e_{k+2} followed by the remaining elements of \mathcal{E} . It is immediately seen that π satisfies the inequalities stated in (3), and hence (\mathcal{E}, C) is a positive instance of CONSCOMP. Conversely, let us assume that (\mathcal{E}, C) is a positive instance of CONSCOMP. Then there exists a permutation $\pi \in S_k$ that satisfies the inequalities stated in (3). Since the first chosen element cannot be negative, it must necessarily be $\frac{m}{2}$. Moreover, since $C = \frac{m}{2}$, the second $\frac{m}{2}$ can be chosen if and only if the energy stored into the gate is zero, that is, if and only if there exists a set $\mathcal{E}' \subseteq \mathcal{E}$ of negative elements whose sum is equal to $-\frac{m}{2}$. The opposites of these elements constitute a set $A' \subseteq A$ such that $\sum_{a \in A'} a = \frac{m}{2}$, and thus we can conclude that A is a positive instance of PARTITION. \square

4. A FRAMEWORK FOR THE STUDY OF ENERGY-BASED PROPERTIES OF COMPUTATIONS

In this section we introduce a framework which can be used to define and study energy-based properties of computations performed by (n, m, d) -gates. The crucial idea of our framework is that we look at computations as a sequence of *conditional movements* of energy. That is, the gate computes its output pattern as follows: for a given subset of input lines, a condition on their values is checked; if this condition is verified then a given action is performed, transforming such values, otherwise no transformation is applied. Successively, another condition is checked on another subset of lines (comprising the output lines from the first step of computation), which determines whether another action has to be performed, and so on until the required values are obtained on the output lines.

To realize the gate according to the aforementioned procedure, we need a (Boolean) *control equipment*, and two *primitives* to conditionally move energy from a given line to another one. We call these primitives *conditional up* (CUP) and *conditional down* (CDOWN). The realization of the gate can thus be viewed as a circuit composed by these simpler elements. Let us first describe CUP and CDOWN as d -valued gates. In the following, we will provide a quantum realization

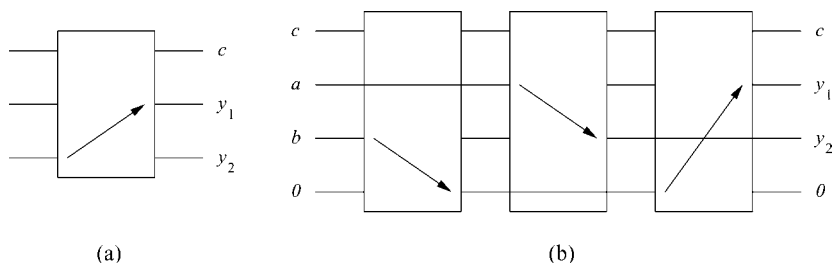


Fig. 1. (a) The conditional up (CUP) gate. (b) Realization of the Boolean Fredkin gate through two-valued CUPs and CDOWNs.

as formulae composed of creation and annihilation operators on \mathbb{C}^d , as we have done for the gates presented in (Cattaneo *et al.*, in press).

The CUP gate is depicted in Fig. 1 (a). It is a $(3, 3, d)$ -gate whose behavior is:

Input: $(c, a, b) \in L_d^3$
 if $c = 1$
 then Output $(c, a + \frac{1}{d-1}, b - \frac{1}{d-1})$
 else Output (c, a, b)

As we can see, c is a control line whose input value is returned unchanged. The condition $c = 1$ enables the movement of a quantity δ of energy from the third to the second line. Of course, this action is performed only if possible, that is, only if $a \neq 1$ and $b \neq 0$ (equivalently, if the energy values associated to the second and third line are not ε_1 and ε_0 , respectively). If these conditions are not satisfied, or if $c \neq 1$, then the gate behaves as the identity. Starting from this description, for any integer $d \geq 2$ we can easily write the truth table of the d -valued CUP gate.

Analogously, the behavior of the complementary $(3, 3, d)$ -gate CDOWN is:

Input: $(c, a, b) \in L_d^3$
 if $c = 1$
 then Output $(c, a - \frac{1}{d-1}, b + \frac{1}{d-1})$
 else Output (c, a, b)

Let us note that $\text{CDown}(c, a, b)$ can be obtained from $\text{CUp}(c, a, b)$ (and vice versa) by exchanging the second and the third line before and after the application of CUP.

Figure 1(b) shows how, using the Boolean versions of CUP and CDOWN gates, we can implement the Boolean Fredkin (controlled switch) gate. Since the Fredkin gate is functionally complete for Boolean logic, using only two-valued CUP and CDOWN gates we can realize any Boolean circuit. In principle, these Boolean circuits, together with d -valued CUPs and CDOWNs, can realize any conditional movement of energy, that is, any conceivable computation that can be performed by (n, m, d) -gates.

It is clear that implementing a gate, be it conservative or not, using only these primitives allows one to visualize the movement of energy between different parts of the gate during a computation. Such visualization may help us to optimize some aspects of the implementation of the gate, namely, the amount of energy moved and the extension of energy jumps. As shown in (Leporati, 2002), such optimizations can be obtained by splitting (if possible) a given (N, M, d) -gate H into k blocks, so that its computation can be performed by an appropriate $(N/k, M/k, d)$ -gate G equipped with a storage unit of capacity C . However, the minimization of the amount of energy moved between different parts of H during the computation is equivalent to the minimization of C , and hence it constitutes an NP-hard problem, whose decision version is the NP-complete problem CONSCOMP. This means that the reorganization of the internal machinery of H to optimize the movements of energy is considered a difficult problem.

Now let us turn to the quantum realization of CUP and CDOWN. Generally, a quantum gate acts on memory cells that are d -level quantum systems called *qudits* (see Cattaneo *et al.*, in press; and Gottesman, 1999). A qudit is typically implemented using the energy levels of an atom or a nuclear spin. The mathematical description—independent of the practical realization—of a single qudit is based on the d -dimensional complex Hilbert space \mathbb{C}^d . In particular, the truth values of L_d are represented by the unit vectors of the canonical orthonormal basis, called the *computational basis* of \mathbb{C}^d :

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \left| \frac{1}{d-1} \right\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots, \quad \left| \frac{d-2}{d-1} \right\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

A collection of n qudits is called a *quantum register* of size n . It is mathematically described by the Hilbert space $\otimes^n \mathbb{C}^d = \underbrace{\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d}_{n \text{ times}}$. An *n-configuration*

is a vector $|x_1\rangle \otimes \dots \otimes |x_n\rangle \in \otimes^n \mathbb{C}^d$, simply written as $|x_1, \dots, x_n\rangle$, for x_i running on L_d . An n -configuration can be viewed as the quantum realization of the “classical” pattern $(x_1, \dots, x_n) \in L_d^n$. Let us recall that the dimension of $\otimes^n \mathbb{C}^d$ is d^n and that the set $\{|x_1, \dots, x_n\rangle : x_i \in L_d\}$ of all n -configurations is an orthonormal basis of this space, called the *n-register computational basis*.

Unlike the situation of the classical wired computer where voltages of a wire go over voltages of another, in quantum realizations of classical gates something different happens. First of all, in this setting every gate must have the same number of input and output lines (that is, they must be (n, n, d) -gates). Each qudit of a given register configuration $|x_1, \dots, x_n\rangle$ (quantum realization of an input

pattern) is in some particular quantum state $|x_i\rangle$ and an operation $G : \otimes^n \mathbb{C}^d \mapsto \otimes^n \mathbb{C}^d$ is performed which transforms this configuration into a new configuration $G(|x_1, \dots, x_n\rangle) = |y_1, \dots, y_n\rangle$, which is the quantum realization of an output pattern. In other words, a quantum realization of an (n, n, d) -gate is a linear operator G that transforms vectors of the n -register computational basis into vectors of the same basis. The action of G on a non-factorized vector, expressed as a linear combination of the elements of the n -register basis, is obtained by linearity.

The collection of all linear operators on \mathbb{C}^d is a d^2 -dimensional linear space whose canonical basis is:

$$\{E_{x,y} = |y\rangle \langle x| : x, y \in L_d\}$$

Since $E_{x,y}|x\rangle = |y\rangle$ and $E_{x,y}|z\rangle = \mathbf{0}$ for every $z \in L_d$ such that $z \neq x$, this operator transforms the unit vector $|x\rangle$ into the unit vector $|y\rangle$, collapsing all the other vectors of the canonical orthonormal basis of \mathbb{C}^d into the null vector. For $i, j \in \{0, 1, \dots, d-1\}$, the operator $E_{\frac{j}{d-1}, \frac{i}{d-1}}$ can be represented as an order d square matrix having 1 in position $(j+1, i+1)$ and 0 in every other position:

$$E_{\frac{j}{d-1}, \frac{i}{d-1}} = (\delta_{r, j+1} \delta_{i+1, s})_{r, s=1, 2, \dots, d}$$

Each of the operators $E_{x,y}$ can be expressed, using the whole algebraic structure of the associative algebra of operators, as a suitable composition of creation and annihilation operators. An alternative approach, that uses spin-creation and spin-annihilation operators, is shown in (Cattaneo *et al.*, in press). We recall that the actions of the *creation* operator a^\dagger and of the *annihilation* operator a on the vectors of the canonical orthonormal basis of \mathbb{C}^d are

$$a^\dagger \left| \frac{k}{d-1} \right\rangle = \sqrt{k+1} \left| \frac{k+1}{d-1} \right\rangle \quad \text{for } k \in \{0, 1, \dots, d-2\}$$

$$a^\dagger |1\rangle = \mathbf{0}$$

and

$$a \left| \frac{k}{d-1} \right\rangle = \sqrt{k} \left| \frac{k-1}{d-1} \right\rangle \quad \text{for } k \in \{1, 2, \dots, d-1\}$$

$$a |0\rangle = \mathbf{0}$$

respectively. Hence, if denote by $A_{u,v}^{p,q,r}$ the expression

$$\underbrace{v \cdots v}_r \underbrace{v^* \cdots v^*}_q \underbrace{v \cdots v}_p u$$

where $u, v \in \{a^\dagger, a\}$, v^* is the adjoint of v , and p, q, r are non negative integer values, then for any $i, j \in \{0, 1, \dots, d - 1\}$ we can express the operator $E_{\frac{i}{d-1}, \frac{j}{d-1}}$ in terms of creation and annihilation as follows:

$$E_{\frac{i}{d-1}, \frac{j}{d-1}} = \begin{cases} \frac{\sqrt{j!}}{(d-1)!} A_{a^\dagger, a^\dagger}^{d-2, d-1-j, 0} & \text{if } i = 0 \\ \frac{\sqrt{j!}}{(d-1)!} A_{a, a^\dagger}^{d-1, d-1-j, 0} & \text{if } i = 1 \text{ and } j \geq 1 \\ \frac{\sqrt{i!}}{(d-1)! \sqrt{j!}} A_{a^\dagger, a^\dagger}^{d-2-i, d-1, j} & \text{if } (i = 1, j = 0 \text{ and } d \geq 3) \text{ or} \\ & (1 < i < d - 2 \text{ and } j \leq i) \\ \frac{\sqrt{j!}}{(d-1)! \sqrt{i!}} A_{a, a}^{i-1, d-1, d-1-j} & \text{if } (i = d - 2, j = d - 1 \text{ and } d \geq 3) \\ & \text{or } (1 < i < d - 2 \text{ and } j > i) \\ \frac{1}{\sqrt{(d-1)! j! (d-1)}} A_{a^\dagger, a}^{d-1, j, 0} & \text{if } i = d - 2 \text{ and } j \leq d - 2 \\ \frac{1}{\sqrt{(d-1)! j!}} A_{a, a}^{d-2, j, 0} & \text{if } i = d - 1 \end{cases}$$

Classical (n, n, d) -gates can be quantistically realized as sums of tensor products of the operators $E_{x,y}$ as follows. Let $x_1 x_2 \dots x_n \mapsto y_1 y_2 \dots y_n$ be a generic row of the truth table of an (n, n, d) -gate. For what we have said earlier, the operator $E_{x_1, y_1} \otimes E_{x_2, y_2} \otimes \dots \otimes E_{x_n, y_n}$ transforms the input configuration $x_1 x_2 \dots x_n$ into the output configuration $y_1 y_2 \dots y_n$, and collapses all the other input configurations of the n -register basis to the null vector. It is not difficult to see that if $\mathcal{O}_0, \dots, \mathcal{O}_{d^n-1}$ are the “local” operators associated to the d^n rows of the truth table, then the operator $\mathcal{O} = \sum_{i=0}^{d^n-1} \mathcal{O}_i$ is a quantum realization of the (n, n, d) -gate. Notice that the resulting operator \mathcal{O} is not necessarily a unitary operator.

Starting from the truth tables of the d -valued gates CUP and CDOWN we can thus build the corresponding linear operators that realize them. For example, it is not difficult to see that the non-unitary linear operator—acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ —which realizes the Boolean CUP gate is:

$$\text{Id} \otimes \text{Id} \otimes \text{Id} - c^\dagger c \otimes aa^\dagger \otimes b^\dagger b + (\text{Id} \otimes a^\dagger \otimes b)(c^\dagger c \otimes aa^\dagger \otimes b^\dagger b) \quad (4)$$

where Id is the identity operator of \mathbb{C}^2 and, for the sake of clearness, we have written $c^\dagger, a^\dagger, b^\dagger$ (resp., c, a, b) to denote the creation (resp., annihilation) operator of \mathbb{C}^2 applied onto the subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ corresponding to the first, second and third input, respectively. In fact, the gate behaves as the identity if the input pattern $|x_c, x_a, x_b\rangle$ is different from $(1, 0, 1)$, since in these cases $(c^\dagger c \otimes aa^\dagger \otimes b^\dagger b)|x_c, x_a, x_b\rangle = \mathbf{0}$, the null vector of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. On the other hand $(c^\dagger c \otimes aa^\dagger \otimes b^\dagger b)|1, 0, 1\rangle = |1, 0, 1\rangle$, hence the first two terms of (4) disappear and the operator $(\text{Id} \otimes a^\dagger \otimes b)$ is applied on $|1, 0, 1\rangle$, giving $|1, 1, 0\rangle$ as required.

In a completely analogous way we can see that the non-unitary linear operator which realizes the Boolean CDOWN gate is:

$$\text{Id} \otimes \text{Id} \otimes \text{Id} - c^\dagger c \otimes a^\dagger a \otimes bb^\dagger + (\text{Id} \otimes a \otimes b^\dagger)(c^\dagger c \otimes a^\dagger a \otimes bb^\dagger)$$

Let us note that the use of creation and annihilation operators allows for different physical implementations. For example, we can view computation not only as a conditional movement of energy but also as a conditional movement of particles between systems that may contain at most $d - 1$ of particles. Alternatively, we can view computation as a sequence of conditional switches of the value of the z -component of the angular momentum of microscopical physical systems, using spin-creation and spin-annihilation instead of creation and annihilation operators (Cattaneo *et al.*, in press).

5. CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

In this paper, we have proposed the first steps towards a theory of conservative computing, where the amount of energy associated to the data which are manipulated during the computations is preserved.

The first obvious extension of our model is to take into account the energy used to perform computations, that is, to transform input values into output values. A first idea is to consider some additional *power source* input lines and *dissipation* output lines. Power source lines are fixed to a constant value from L_d (usually 1), and absorb energy from the environment. This energy is entirely consumed during the computation, whereas all the energy associated to the input pattern is returned by the output pattern. On the other hand, dissipation lines are used to model the release of energy into the environment; hence, their value is simply discarded. Conservative gates constitute a special case in our framework, where there are neither power source nor dissipation lines (under the hypothesis that we do not take into account the energy needed to perform the computation).

Since perfect conservation of energy can be obtained only in theory, a second possibility for future work could be to relax the conservativeness constraint (2), by assuming that the amount of energy dissipated during a computation step is not greater than a fixed value. Analogously, we can suppose that if we try to store an amount of energy that exceeds the capacity of the gate then the energy which cannot be stored is dissipated. In such a case, it should be interesting to study trade-offs between the amount of energy dissipated and the hardness of the corresponding modified CONSCOMP problem.

Finally, it remains to study how to theoretically model and physically realize gates equipped with an internal storage unit. Here we just observe that, from a theoretical point of view, it seems appropriate to consider this kind of gates as

finite state automata, by viewing the energy levels of the storage unit as their states.

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